A NEW BEAM FINITE ELEMENT WITH SEVEN DEGREES OF FREEDOM
AT EACH NODE FOR THE STUDY OF COUPLED BENDING-TORSION VIBRATIONS

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1. INTRODUCTION

The subject matter of the first part of this study is the derivation of coupled bending-torsion relations characterizing the dynamical behaviour of unsymetrical cross-section beams. This allows for the further definition of a beam element with seven degrees of freedom per node. The numerical results obtained with the FEM are compared to experiment in some section shape cases.

In order to characterize the displacement field of considered beams, the method of integrated displacements allows us to consider the so-called secondary effects: longitudinal warping inertia, and shear deformation due to both shearing forces and nonuniform warping. The literature on dynamical flexure and torsion of beams is extensive. Cowper [1] for flexure and Gay-Boudet [2] for torsion introduce integrated displacements. We extend in the following this notion to the study of coupled bending-torsion of an homogeneous straight beam.

2. BASIC THEORY

2.1 Displacement Field

Ω is the domain occupied by the cross-section. Γ is the boundary, and G is the centroid. In the plane of the section, the principal axis are noted Gx_2, Gx_3. C (c_2, c_3) is the shear center, as defined by Trefftz [3] (figure 1).

The displacement of any point M of the beam is noted $\vec{X}_M(x_i)$, with components X_i (i=1,2,3). Let us define the seven displacement parameters:

Three angular parameters: $\quad \theta_i(x_1,t) = \dfrac{1}{I_i}\displaystyle\int_\Omega (\vec{GM} \times \vec{X}_M).\vec{x}_i \ d\Omega \qquad$ (1-a)

Three linear parameters: $\quad U_i(x_1,t) = \dfrac{1}{S}\displaystyle\int_\Omega \vec{X}_M.\vec{x}_i \ d\Omega \qquad$ (1-b)

A warping parameter: $\quad \overset{\circ}{\theta}(x_1,t) = \dfrac{1}{I_\Phi}\displaystyle\int_\Omega \Phi.X_1 \ d\Omega \qquad$ (1-c)

$\Phi(x_2,x_3)$ is the Saint-Venant warping function defined in C, and $I_\Phi = \displaystyle\int_\Omega \Phi^2 \ d\Omega$ is the quadratic warping moment.

$$\vec{X} = \vec{r} + \vec{GM} \times \vec{\theta} + \Phi\dot{\theta}\vec{x}_1 + \vec{\eta} \tag{2}$$

According to the above definitions, necessary orthogonality conditions hold in the domain Ω for the functions 1, x_2, x_3, Φ, η_1, together with:

$$\int_\Omega \eta_2 \ d\Omega = \int_\Omega \eta_3 \ d\Omega = \int_\Omega (x_2\eta_3 - x_3\eta_2) \ d\Omega = 0 \tag{3}$$

2.2 Equations of Motion

Assuming the lateral surface of the beam free of any force, we take into account as first hypothesis (H.1) the assumption that normal stresses σ_{22} and σ_{33} are expected to be negligible compared to σ_{11}. Then, the principle of virtual work associated with the displacement field (2) leads to the classical set of motion equations:

$$\rho S\vec{U},_{tt} = \vec{F},_1 + \vec{p} \tag{4-a,b,c}$$

$$\rho[1]\vec{\theta},_{tt} = \vec{M},_1 + \vec{m} + \vec{x}_1 \times \vec{F} \tag{5-a,b,c}$$

And the seventh motion equation is the bimoment one:

$$\rho I_\Phi \dot{\theta},_{tt} = B,_1 + b + (M_1 - GJ\theta_1,_1) - c_2 F_3 + c_3 F_2 \tag{6}$$

in which $B = \int_\Omega \Phi\sigma_{11} \ d\Omega$ is the generalized bimoment.

2.3 Constitutive Equations

In order to obtain a technical formulation for the constitutive equations, we assume as second hypothesis (H.2) that the change in the deformation $\vec{\eta}$ of two infinitely near adjacent cross sections is neglected in order to evaluate the longitudinal and shear stresses σ_{1j}, $j = 1,2,3$. Thus, starting from the Hooke's law for a linear elastic body, the constitutive equations below are deduced from integrations over the domain of the section:

$$u_1,_1 = \frac{F_1}{ES} \quad ; \quad \theta_2,_1 = \frac{M_2}{EI_2} \quad ; \quad \theta_3,_1 = \frac{M_3}{EI_3} \quad ; \quad \dot{\theta},_1 = \frac{B}{EI_\Phi} \tag{7-a,b,c,d}$$

Moreover, noting that the first component η_1 of $\vec{\eta}$ (2) is essentially due to the shear forces F_2, F_3 and nonuniform warping moment noted $M_{nuw} = (M_1 - GJ\theta_1,_1)$, we obtain the three followed coupled relations written in matrix form:

$$\begin{bmatrix} \theta_1,_1 - \dot{\theta} \\ u_2,_1 - \theta_3 - c_3\dot{\theta} \\ u_3,_1 + \theta_2 + c_2\dot{\theta} \end{bmatrix} = \begin{bmatrix} K_{11} & -K_{12} & -K_{13} \\ -K_{21} & K_{22} & -K_{23} \\ -K_{31} & -K_{32} & K_{33} \end{bmatrix} \begin{bmatrix} M_{nuw}/G(I_1-J) \\ F_2/GS \\ F_3/GS \end{bmatrix} \tag{8-a,b,c}$$

In (8) appear nine shear coefficients with the symmetry proper-ties:

$$K_{23} = K_{32} \qquad ; \qquad K_{1j} = K_{j1} \frac{I_1-J}{S} \ , \ j=2,3 \qquad (9\text{-}a,b,c)$$

Starting from the local equilibrium of continuous media, we show that the shear coefficients K_{ij} can be computed after solving three Poisson's problems in the cross section, namely:

$$\begin{cases} \nabla^2 g = f(x_2,x_3) \text{ over } \Omega, \text{ with successively } f = x_2,x_3,\Phi. \\ \\ \partial g/\partial n = 0 \text{ along the boundary } \Gamma \qquad\qquad (10\text{-}a,b,c) \end{cases}$$

$$\text{with } \int_\Omega g \ d\Omega = 0$$

For the complete expressions of K_{ij}, see appendix A. The effective calculus of all bending-torsion constants involved in the aforemen-tioned relations (I_i, J, I_Φ, K_{ij}, c_i) has been performed by means of a boundary element method [4].

3.   FINITE ELEMENT FORMULATION

For the setting up of a finite element formulation, let us first look at the technical expressions of potential and kinetic energies.

3.1   Potential Strain Energy $V(x_1,t)$

For a beam element of length dx_1, and according to H.1, the general form of strain energy:

$$dV = \{\frac{1}{2} \int_\Omega \sigma_{ij}\varepsilon_{ij}d\Omega\}dx_1$$

reduces to:

$$V,_1 = \frac{1}{2} \int_\Omega (\sigma_{11}\varepsilon_{11} + 2\sigma_{12}\varepsilon_{12} + 2\sigma_{13}\varepsilon_{13}) \ d\Omega \qquad (11)$$

Consideration of (2) and H.2 leads to the practical expressions:

$$V,_1 = V_\sigma,_1 + V_\tau,_1 \qquad (12)$$

in which:

$$V_\sigma,_1 = \frac{1}{2} \{\frac{F_1^2}{ES} + \frac{M_2^2}{EI_2} + \frac{M_3^2}{EI_3} + \frac{B^2}{EI_\Phi}\} \qquad (13\text{-}a)$$

$$V_\tau,_1 = \frac{1}{2} \{K_{22} \frac{F_2^2}{GS} + K_{33} \frac{F_3^2}{GS} + K_{11} \frac{M_{nuw}^2}{G(I_1-J)} \ \cdots$$

$$\cdots + \frac{(GJ\theta_1,_1)^2}{G} - (K_{23}+K_{32})\frac{F_2 F_3}{GS} - \frac{M_{nuw}}{G(I_1-J)} \{F_2 (K_{12}+\frac{I_1-J}{S} K_{21}) + \cdots$$

$$\ldots \; F_3(K_{13}+\frac{I_1-J}{S} K_{31})\} \qquad (13\text{-}b)$$

3.2 Kinetic Energy

In the same way, we retain the simple expression:

$$T_{,1} = \frac{1}{2} \rho S(\vec{U},_t)^2 + \frac{1}{2} \rho I_\phi \overset{\circ}{\theta}{}^2,_t + \frac{1}{2} \rho\{\vec{\theta},_t\}^T |I| \{\vec{\theta},_t\} \qquad (14)$$

3.3 Definition of the Element

Seven displacements per node k characterize the motion of the two-nodes beam element. In a matricial form:

$$\{W_k\} = [u_1,u_2,\theta_3,u_3,\theta_2,\theta_1,\overset{\circ}{\theta}]^T_k$$

Substituting the static case for the quasi-static dynamic one leads to an interpolation matrix $[A]$ not detailed here, allowing a displacement field in the form of:

$$\{W(x_1,t)\} = [A(x_1)]\{\overset{\bullet}{W}(t)\} \qquad (15)$$

$$\text{where} \quad \{\overset{\bullet}{W}(t)\} = \begin{Bmatrix} W_1(t) \\ W_2(t) \end{Bmatrix}$$

3.4 Stiffness Matrix

Introducing the displacement field (15), the element strain energy is obtained after integration of (13) over the length ℓ of the element, and written in the matrix form:

$$2V = \{\overset{\bullet}{W}\}^T[D]\{\overset{\bullet}{W}\}$$

We detail in Appendix B the stiffness matrix $[D]$ in the simplified case $i{\neq}j{\rightarrow}K_{ij}{\sim}0$. It can be noted that, for symetrical section shape cases and for uniform warping (Saint-Venant torsion), this stiffness matrix reduces to the classical one derived by Przemieniecki [5].

3.5 Mass Matrix

The calculus of the kinetic energy for the whole element by means of (15) leads to the form:

$$2T = \{\overset{\bullet}{W},_t\}^T[M]\{\overset{\bullet}{W},_t\}$$

in which $[M]$ is a consistent mass matrix, reached after very heavy calculus. In the much more simpler case of lumped mass approximation, the kinetic energy reduces to:

$$2T = \rho \frac{\ell}{2} \sum_{k=1,2} \{ \sum_{j=1,2,3} (Su_{kj,t}^2 + I_j\theta_{kj,t}^2) + I_\phi\dot{\theta}_{k,t}^2\}$$

from which a simple diagonal mass-matrix is easily derived.

4. APPLICATIONS

Numerical and experimental tests have been performed on several cantilever beams. For the dynamical flexure and torsion of a rectangular section beam, numerical values of frequencies provided by FEM and analytical ones are in good agreement with the experiment, especially for high frequencies. The dynamical torsion of an I-section beam has been likewise investigated. In such a case, exact analytical values of torsional constants cannot be reached, needing a previous computational work. The results for dynamical case are detailed in reference [6], and some of them are recalled in appendix C. In a same manner, we have also tested two U-section beams (thick and thin). Here coupling between flexure and torsion occurs, and the whole theory above applies. We shall present both numerical and experimental frequencies. Rather good agreement can be noted for all tests, the rank of the modes concerned depending on the number of elements of the discretization.

5. CONCLUSION

The study of coupled bending-torsion can be performed with acceptable accuracy by means of the formulation above, starting from the definition of seven displacements parameters in each section of the beam. The finite element derived allows a simple numerical prediction of the dynamical behaviour of beams with any cross sections. Nevertheless, we must keep in mind that an accurate computation of coupled bending-torsion constants is the first stage when using this element.

NOMENCLATURE

E,G   Young's and shear modulus.

S, I_1, I_2, I_3 Cross section area and quadratic moments of inertia in G.

I_ϕ Quadratic warping moment.

J   Saint-Venant's torsional rigidity.

F_1, F_2, F_3 Normal and shear forces.

M_1, M_2, M_3 Torsion torque and bending moments.

$\vec{U}, \vec{\theta}$ Linear and angular displacement vectors (components: u_i and θ_i).

$\vec{x}_i$ Unit vectors of principal axis.

ρ Mass per unit volume.

$( )_{,i}$ Partial derivative with respect to x_i.

$( )_{,t}$ id. with time t.

$\partial/\partial n$ Outward normal derivative along the boundary Γ.

$\vec{p}, \vec{m}, b$ Distributed forces, moments and bimoment along the beam.

$[I]$ Diagonal inertia matrix of cross section.

$\vec{\eta}$ Complementary displacement vector.

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Appendix A

Shear Coefficients K_{ij}

Solutions of (10-a,b,c) allow the knowledge of the functions $g_{i_0}(i=1,2,3)$ verifying over the domain Ω of the cross-section:

$$\nabla^2 g_{10} = \frac{S c_3}{I_\Phi} \Phi - \frac{S}{I_3} x_2$$

$$\nabla^2 g_{20} = -\frac{S c_2}{I_\Phi} \Phi - \frac{S}{I_2} x_3$$

$$\nabla^2 g_{30} = \frac{I_1 - J}{I_\Phi} \Phi$$

with $\partial g_{i_0}/\partial n = 0$; $i=1,2,3$ along the boundary Γ.

Then, the K_{ij} are deduced in the form of:

$$K_{11} = -\frac{1}{I_\Phi} \int_\Omega \Phi g_{30} \, d\Omega$$

and for $i,j=2,3$; $i \neq j$; $k=j-1$:

$$K_{jj} = \frac{1}{I_i} \int_\Omega x_j \, g_{k_0} \, d\Omega + (j-i) \frac{c_i}{I_\phi} \int_\Omega \Phi g_{k_0} \, d\Omega$$

$$K_{ji} = -\frac{1}{I_j} \int_\Omega x_j \, g_{k_0} \, d\Omega + (j-i) \frac{c_j}{i_\phi} \int_\Omega \Phi g_{k_0} \, d\Omega$$

$$K_{j1} = \frac{I_1-J}{SI_\phi} \int_\Omega \Phi g_{k_0} \, d\Omega$$

$$\frac{I_1-J}{S} K_{1j} = -\frac{1}{I_i} \int_\Omega x_j \, g_{30} \, d\Omega + (i-j) \frac{c_i}{I_\phi} \int_\Omega \Phi g_{30} \, d\Omega$$

The properties of symetry of the K_{ij} are shown by means of the Green-Gauss theorem, and we find:

$$K_{23} = K_{32} \quad ; \quad K_{j1} = \frac{I_1-J}{S} K_{1j}$$

Appendix B

Stiffness Matrix $[D]$

We introduce the following dimensionless notations:

$$\alpha = 1 + K_{11} \frac{J}{I_1-J} \quad ; \quad \lambda^2 \ell^2 = \frac{GJ\ell^2}{\alpha EI_\phi} \quad ; \quad k_1 = \frac{\lambda\ell \, sh\lambda\ell}{ch\lambda\ell-1}$$

$$I_\phi k_0 = \left\{ \frac{1}{\Phi_1} (\alpha k_1-2) + \frac{\alpha}{I_\phi} \left(\frac{I_3 c_3^2}{1+\Phi_3} + \frac{I_2 c_2^2}{1+\Phi_2} \right) (k_1-2) \right\}^{-1}$$

$$\Phi_1 = 12 K_{11} \frac{EI_\phi}{G(I_1-J)\ell^2} \quad ; \quad \beta_1 = \alpha \frac{k_0 I_\phi}{\Phi_1} (2-k_1)$$

$$j=2,3 \rightarrow \Phi_j = 12 K_{jj} \frac{EI_j}{GS\ell^2} \quad ; \quad \beta_j = \alpha k_0 \frac{I_j c_j^2}{1+\Phi_j} (2-k_1)$$

Then, the symetry properties of $[D]$ leads to the following terms, in the simplified case : $i \neq j \rightarrow K_{ij}=0$.

$$d_1^1 = d_8^8 = -d_8^1 = -d_1^8 = \frac{ES}{\ell} \quad ; \quad \text{others } d_j^1 = d_1^j = d_j^8 = d_8^j = 0$$

$$d_2^2 = d_9^9 = -d_9^2 = \frac{12 \, EI_3}{\ell^3(1+\Phi_3)} (1+\beta_3)$$

$$d_3^2 = d_{10}^2 = -d_{10}^9 = -d_9^3 = \frac{\ell}{2} d_2^2$$

$$d_4^2 = d_{11}^9 = -d_{11}^2 = -d_9^4 = -\frac{12\, EI_3}{\ell^3(1+\Phi_3)}\, \beta_2\, c_3/c_2 = -\frac{12\, EI_2}{\ell^3(1+\Phi_2)}\, \beta_3\, c_2/c_3$$

$$d_5^2 = d_{12}^2 = -d_9^5 = -d_{12}^9 = -d_4^3 = -d_{10}^4 = d_{11}^3 = -\frac{\ell}{2} d_4^2$$

$$d_6^2 = d_{13}^9 = -d_{13}^2 = -d_9^6 = \frac{12\, EI_3}{\ell^3(1+\Phi_3)}\, \beta_1\, c_3$$

$$d_7^2 = d_{14}^2 = -d_9^7 = -d_{14}^9 = k_0\, GJ\, \frac{I_3\, c_3}{1+\Phi_3}$$

$$d_3^3 = d_{10}^{10} = \frac{EI_3}{\ell(1+\Phi_3)}\, (4+\Phi_3+3\beta_3)$$

$$d_5^3 = d_{12}^3 = d_{12}^{10} = -\frac{\ell}{2} d_4^3$$

$$d_6^3 = -d_{13}^3 = d_{10}^6 = -d_{13}^{10} = \frac{\ell}{2} d_6^2$$

$$d_7^3 = d_{14}^3 = d_{10}^7 = d_{14}^{10} = \frac{\ell}{2} d_7^2$$

$$d_{10}^3 = \frac{EI_3}{\ell(1+\Phi_3)}\, (2-\Phi_3+3\beta_3)$$

$$d_4^4 = d_{11}^{11} = -d_{11}^4 = \frac{12\, EI_2}{\ell^3(1+\Phi_2)}\, (1+\beta_2)$$

$$d_5^4 = d_{12}^4 = -d_{12}^{11} = -d_{11}^5 = -\frac{\ell}{2} d_4^4$$

$$d_6^4 = d_{13}^{11} = -d_{13}^4 = -d_{11}^6 = -\frac{12\, EI_2}{\ell^3(1+\Phi_2)}\, \beta_1\, c_2$$

$$d_7^4 = d_{14}^4 = d_{11}^7 = -d_{14}^{11} = -k_0\, GJ\, \frac{I_2\, c_2}{1+\Phi_2}$$

$$d_5^5 = d_{12}^{12} = \frac{EI_2}{\ell(1+\Phi_2)}\, (4+\Phi_2+3\beta_2)$$

$$d_6^5 = -d_{13}^{12} = -d_{13}^5 = d_{12}^6 = -\frac{\ell}{2} d_6^4$$

$$d_7^5 = d_{14}^5 = d_{14}^{12} = d_{12}^7 = -\frac{\ell}{2} d_7^4$$

$$d_{10}^5 = -\frac{\ell}{2} d_{10}^4$$

$$d_{12}^5 = \frac{EI_2}{\ell(1+\Phi_2)}\, (2-\Phi_2+3\beta_2)$$

$$d_6^6 = d_{13}^6 = d_{13}^{13} = \beta_1\, \frac{GJ}{\ell}\, \frac{k_1}{2-k_1} - 12\alpha\, \frac{EI_\Phi}{\ell^3\Phi_1}\, (\beta_2+\beta_3)$$

$$d_7^6 = d_{14}^6 = -d_{13}^7 = -d_{14}^{13} = k_0 \; I_\Phi \; \frac{GJ}{\Phi_1}$$

$$d_7^7 = d_{14}^{14} = -\frac{GJk_0}{\alpha\lambda(ch\lambda\ell-1)} \left\{\frac{I_\Phi}{\Phi_1} (sh\lambda\ell - \alpha\lambda\ell ch\lambda\ell) + \alpha \left(\frac{I_2 \; c_2^2}{1+\Phi_2} + \frac{I_3 \; c_3^2}{1+\Phi_3}\right) \ldots\right.$$

$$\ldots (sh\lambda\ell - \lambda\ell ch\lambda\ell) \}$$

$$d_{14}^7 = \frac{GJk_0}{\alpha\lambda(ch\lambda\ell-1)} \left\{\frac{I_\Phi}{\Phi_1} (sh\lambda\ell - \alpha\lambda\ell) + \alpha \left(\frac{I_2 \; c_2^2}{1+\Phi_2} + \frac{I_3 \; c_3^2}{1+\Phi_3}\right)(sh\lambda\ell - \lambda\ell)\right\}$$

Appendix C

The study of natural torsional frequencies of a cantilever beam performed by means of the finite element procedure has been compared to the experimental datas. The results concerning the relative error are shown on table 1 below.
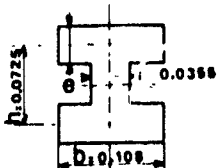
MODE N°	1	2	3	4	5	6	7	8	9	10
ERROR %	0.	0.9	0.	0.7	-1.	-0.9	-1.6	-1.3	0.1	2.

Table 1.

Figure 1.